

ON THE SUPPORT OF MEASURES IN MULTIPLICATIVE FREE CONVOLUTION SEMIGROUPS

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ABSTRACT. In this paper, we study the supports of measures in multiplicative free semigroups on the positive real line and on the unit circle. We provide formulas for the density of the absolutely continuous parts of measures in these semigroups. The descriptions rely on the characterizations of the images of the upper half-plane and the unit disc under certain subordination functions. These subordination functions are η -transforms of infinitely divisible measures with respect to multiplicative free convolution. The characterizations also help us study the regularity properties of these measures. One of the main results is that the number of components in the support of measures in the semigroups is a decreasing function of the semigroup parameter.

1. INTRODUCTION

Denote by $\mathcal{M}_{\mathbb{R}_+}$ and $\mathcal{M}_{\mathbb{T}}$ the set of Borel probability measures on the positive real line $\mathbb{R}_+ = [0, \infty)$ and on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, respectively. Further let $\mathcal{M}_{\mathbb{R}_+}^\times = \mathcal{M}_{\mathbb{R}_+} \setminus \{\delta_0\}$, and let $\mathcal{M}_{\mathbb{T}}^\times$ be the subset of $\mathcal{M}_{\mathbb{T}}$ consisting of measures whose first moments are nonzero and η -transforms (see section 2 for definition) never vanish throughout $\mathbb{D} \setminus \{0\}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For measures μ and ν both in $\mathcal{M}_{\mathbb{R}_+}^\times$ or in $\mathcal{M}_{\mathbb{T}}^\times$, their multiplicative free convolution, denoted by $\mu \boxtimes \nu$, can be characterized via the Σ -transforms (see section 2 for definition). That is, the identity $\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$ holds for z in some appropriate region. We refer the reader to [10, 27] for more details.

One of the significant properties of multiplicative free convolution is the existence of subordination functions. More precisely, the η -transform $\eta_{\mu \boxtimes \nu}$ of measures μ and ν either both in $\mathcal{M}_{\mathbb{R}_+}$ or in $\mathcal{M}_{\mathbb{T}}$ is subordinated to η_μ in the sense that $\eta_{\mu \boxtimes \nu} = \eta_\mu \circ \omega$ for some unique analytic function ω . The function ω is a self-mapping of $\mathbb{C} \setminus \mathbb{R}_+$ if $\mu, \nu \in \mathcal{M}_{\mathbb{R}_+}$ while it is a self-mapping of \mathbb{D} if $\mu, \nu \in \mathcal{M}_{\mathbb{T}}$ [5, 15, 25]. One of the applications of subordination functions is the study the regularity of multiplicative free convolution. This fact, first noted by Voiculescu [25], has been widely used in free probability theory. For instance, if λ_t , $t \geq 0$, is the multiplicative free Brownian motion on \mathbb{R}_+ or \mathbb{T} then $\eta_{\mu \boxtimes \lambda_t} = \eta_\mu \circ \omega_t$ for some analytic function ω_t whose η -transform is a \boxtimes -infinitely divisible measure. It turns out that the measure $\mu \boxtimes \lambda_t$ is absolutely continuous for any $t > 0$, and its density can be described in terms of ω_t . More importantly, the number of components in the support of $\mu \boxtimes \lambda_t$ is a decreasing function of t . We refer

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the reader to [14,29] for more details. The same conclusion also holds for additive free convolution [13].

For $n \in \mathbb{N}$, and μ in $\mathcal{M}_{\mathbb{R}_+}$ or $\mathcal{M}_{\mathbb{T}}$, the n -fold multiplicative free convolution $\mu \boxtimes \cdots \boxtimes \mu$ is denoted by $\mu^{\boxtimes n}$. The measure $\mu^{\boxtimes n}$ can be characterized by its Σ -transform, that is, the identity $\Sigma_{\mu^{\boxtimes n}} = \Sigma_{\mu}^n$ holds in some appropriate region. This can be generalized to any convolution power $t \geq 1$. More precisely, if $\mu \in \mathcal{M}_{\mathbb{R}_+}^{\times}$ (resp. $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$) and $t \geq 1$ there exists a semigroup $\{\mu_t : t \geq 1\}$ contained in $\mathcal{M}_{\mathbb{R}_+}^{\times}$ (resp. contained in $\mathcal{M}_{\mathbb{T}}^{\times}$) such that the identity $\Sigma_{\mu_t} = \Sigma_{\mu}^t$ holds in some appropriate domain, where the t th power is taken appropriately. The measure μ_t coincides with $\mu^{\boxtimes n}$ if $t = n$ is an integer. With the help of the subordination in this context, it was shown that the measure μ_t has no singular continuous part and its density is analytic wherever it is positive [3,4].

In this paper, we study the supports of measures in the semigroup $\{\mu_t : t \geq 1\}$ associated with the measure μ which in $\mathcal{M}_{\mathbb{R}_+}^{\times}$ or in $\mathcal{M}_{\mathbb{T}}^{\times}$. The main tools used in this study are the properties of subordination functions established in [4]. By the methods developed in this paper, we show that the number of components in the support of μ_t is a decreasing function of t . The corresponding subordination functions are shown to be the η -transforms of \boxtimes -infinitely divisible measures and their ranges are also analyzed. Another purpose of this paper is to give an implicit formula for the density of the absolutely continuous part of μ_t .

The paper is organized as follows. Section 2 contains some definitions and preliminaries. Section 3 and Section 4 investigate the topological properties of measures in the semigroups associated with measures μ on the positive real line and the unit circle, respectively.

2. PRELIMINARY

Following the notation in the introduction, the ψ -transform of μ is defined as

$$\psi_{\mu}(z) = \int \frac{zs}{1-zs} d\mu(s),$$

which is analytic on $\Omega = \mathbb{C} \setminus [0, +\infty)$ if $\mu \in \mathcal{M}_{\mathbb{R}_+}$ and analytic on \mathbb{D} if $\mu \in \mathcal{M}_{\mathbb{T}}$. The η -transform of μ is defined as

$$\eta_{\mu} = \frac{\psi_{\mu}}{1 + \psi_{\mu}}$$

on the same domain as the ψ -transform. The analytic way to obtain the multiplicative free convolution is by using the inverse of the η -transform. A measure μ is said to be \boxtimes -infinitely divisible if for any integer n there exists a measure μ_n such that

$$\mu = \underbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}_{n \text{ factors}}.$$

We refer the reader to [9,10] for more information about multiplicative free convolution and \boxtimes -infinite divisibility. In the following two subsections, we briefly introduce some

properties of these transforms and the multiplicative free semigroups associated with a measure in $\mathcal{M}_{\mathbb{R}_+}^\times$ or $\mathcal{M}_{\mathbb{T}}^\times$.

2.1. Multiplicative Free Convolution Semigroup on \mathbb{R}_+ . Measures in $\mathcal{M}_{\mathbb{R}_+}^\times$ can be characterized by their η -transforms, which is stated in the following proposition.

Proposition 2.1. *Let $\eta : \Omega \rightarrow \mathbb{C} \setminus \{0\}$ be an analytic function such that $\eta(\bar{z}) = \overline{\eta(z)}$ for all $z \in \Omega$. Then the following statements are equivalent.*

- (1) *There exists a measure $\mu \in \mathcal{M}_{\mathbb{R}_+}^\times$ such that $\eta = \eta_\mu$.*
- (2) *The function η satisfies $\eta(0-) = 0$ and $\arg \eta(z) \in [\arg z, \pi]$ for $z \in \mathbb{C}^+$.*

Any measure $\mu \in \mathcal{M}_{\mathbb{R}_+}$ can be recovered from its η -transform by Stieltjes inversion formula. Indeed, the identity

$$G_\mu\left(\frac{1}{z}\right) = \frac{z}{1 - \eta_\mu(z)}, \quad z \in \Omega,$$

where G_μ is the Cauchy transform of μ , shows that the family of measures $\{\mu_\epsilon\}_{\epsilon > 0}$ defined as

$$d\mu_\epsilon(1/x) = \frac{1}{\pi} \Im \left(\frac{x + i\epsilon}{1 - \eta_\mu(x + i\epsilon)} \right) dx, \quad x > 0,$$

converges to ν weakly as $\epsilon \rightarrow 0$, where $d\nu(x) = d\mu(1/x)$. Note that ν is not always in $\mathcal{M}_{\mathbb{R}_+}$ since $\nu(\mathbb{R}_+) = 1 - \mu(\{0\})$.

If $\mu \in \mathcal{M}_{\mathbb{R}_+}^\times$ then $\eta'_\mu(z) > 0$ for $z < 0$, and therefore $\eta_\mu|_{(-\infty, 0)}$ is invertible. Let η_μ^{-1} be the inverse of η_μ and set

$$\Sigma_\mu(z) = \frac{\eta_\mu^{-1}(z)}{z},$$

where $z < 0$ is sufficiently small. For $n \in \mathbb{N}$, the multiplicative free convolution power $\mu^{\boxtimes n}$ of μ satisfies the identity

$$\Sigma_{\mu^{\boxtimes n}}(z) = \Sigma_\mu^n(z),$$

where $z < 0$ is sufficiently small. The generalization to any multiplicative free convolution power greater than one is stated below [4].

Theorem 2.2. *If $\mu \in \mathcal{M}_{\mathbb{R}_+}^\times$ and $t > 1$ then the following statements hold.*

- (1) *There exists a unique measure $\mu^{\boxtimes t} \in \mathcal{M}_{\mathbb{R}_+}^\times$ such that $\Sigma_{\mu^{\boxtimes t}}(z) = \Sigma_\mu^t(z)$ for $z < 0$ sufficiently close to zero.*
- (2) *There exists an analytic function $\omega_t : \Omega \rightarrow \Omega$ such that $\omega_t((-\infty, 0)) \subset (-\infty, 0)$, $\omega_t(0-) = 0$, $\arg \omega_t(z) \in [\arg z, \pi]$ for $z \in \mathbb{C}^+$, and $\eta_{\mu^{\boxtimes t}} = \eta_\mu \circ \omega_t$.*
- (3) *The function $\Phi_t : \Omega \rightarrow \mathbb{C} \setminus \{0\}$ defined by*

$$\Phi_t(z) = z \left[\frac{z}{\eta_\mu(z)} \right]^{t-1}, \quad z \in \Omega,$$

where the power is taken to positive for $z < 0$, satisfies $\Phi_t(\omega_t(z)) = z$ for $z \in \Omega$.

- (4) Let $x > 0$. Then the point $1/x$ is an atom of $\mu^{\boxtimes t}$ if $\mu(\{x^{-1/t}\}) > (t-1)/t$, in which case we have

$$\mu^{\boxtimes t}(1/x) = t\mu(\{x^{-1/t}\}) - (t-1).$$

Next, we introduce some special mappings and sets which connect Theorem 2.2 and the global inversion theorem.

Denote by \mathcal{S}_π the strip $\{z \in \mathbb{C} : |\Im z| < \pi\}$. Further, let $\mathcal{S}_\pi^+ = \mathcal{S}_\pi \cap \mathbb{C}^+$ and $\mathcal{S}_\pi^- = \mathcal{S}_\pi \cap \mathbb{C}^-$. The map $\Lambda(z) = -e^z : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ is a conformal mapping from \mathcal{S}_π onto Ω . Particularly, we have $\Lambda(\mathcal{S}_\pi^+) = \mathbb{C}^-$ and $\Lambda(\mathcal{S}_\pi^-) = \mathbb{C}^+$. If Λ^{-1} is the inverse of this conformal mapping then

$$(2.1) \quad \Im \Lambda^{-1}(z) = -\pi + \arg z, \quad z \in \mathbb{C}^+$$

and

$$(2.2) \quad \Im \Lambda^{-1}(z) = \pi + \arg z, \quad z \in \mathbb{C}^-.$$

For any measure $\mu \in \mathcal{M}_{\mathbb{R}_+}^\times$, by Proposition 2.1 we have

$$(2.3) \quad \arg[\eta_\mu(\Lambda(z))] \in (-\pi, -\pi + \Im z], \quad z \in \mathcal{S}_\pi^+$$

and

$$(2.4) \quad \arg[\eta_\mu(\Lambda(z))] \in [\pi + \Im z, \pi), \quad z \in \mathcal{S}_\pi^-.$$

Moreover, the function

$$\kappa_\mu(z) = \frac{z}{\eta_\mu(z)}$$

is analytic on Ω since η_μ never vanishes on Ω . For any $t > 1$, let

$$(2.5) \quad H_t(z) = z + (t-1)l_\mu(z), \quad z \in \mathcal{S}_\pi,$$

where

$$(2.6) \quad l_\mu(z) = \Lambda^{-1}[-\kappa_\mu(\Lambda(z))].$$

Next, observe that

$$(2.7) \quad \Im H_t(z) \in [\Im z, t\Im z), \quad z \in \mathcal{S}_\pi^+$$

or, equivalently, $\Im l_t(z) \in [0, \Im z)$, $z \in \mathcal{S}_\pi^+$. Indeed, if $z \in \mathcal{S}_\pi^+$ then

$$\arg \kappa_\mu(\Lambda(z)) = \arg \Lambda(z) - \arg \eta_\mu(\Lambda(z)) \in [0, \Im z)$$

by (2.3), which yields $\arg[-\kappa_\mu(\Lambda(z))] \in [-\pi, -\pi + \Im z)$ and $-\kappa_\mu(\Lambda(z)) \in \mathbb{C}^-$. Hence by (2.2) we have

$$\Im \Lambda^{-1}[-\kappa_\mu(\Lambda(z))] = \pi + \arg[-\kappa_\mu(\Lambda(z))] \in [0, \Im z),$$

as desired.

The following theorem, obtained by choosing $h = \pi$ and $k = t\pi$ in [Theorem 4.9, 4], plays an important role in the investigation of the support of $\mu^{\boxtimes t}$.

Theorem 2.3. *If H_t is the analytic function defined in (2.5) then the following statements hold.*

- (1) *There exists an analytic function $\varpi_t : \mathcal{S}_\pi \rightarrow \mathcal{S}_\pi$ such that ϖ_t extends continuously to $\overline{\mathcal{S}_\pi}$, $|\Im \varpi_t(z)| \leq |\Im z|$, $\varpi_t(\overline{z}) = \overline{\varpi_t(z)}$, and $H_t(\varpi_t(z)) = z$ for all $z \in \mathcal{S}_\pi$.*
- (2) *The function ϖ_t satisfies*

$$\frac{|z_1 - z_2|}{2(t+1)} \leq |\varpi_t(z_1) - \varpi_t(z_2)|, \quad z_1, z_2 \in \overline{\mathcal{S}_\pi}.$$

- (3) *The set $\{z \in \mathcal{S}_\pi : H_t(z) \in \mathcal{S}_\pi\}$ coincides with $\varpi_t(\mathcal{S}_\pi)$ and is a simply connected domain whose boundary consists of two simple curves, $\varpi_t(\mathbb{R} \pm i\pi)$.*
- (4) *If $\alpha \in \partial \mathcal{S}_\pi$ and $\varpi_t(\alpha) \in \mathcal{S}_\pi$ then ϖ_t can be continued analytically to a neighborhood of α .*

2.2. Multiplicative Free Convolution Semigroup on \mathbb{T} . The following proposition characterizes functions which are η -transforms.

Proposition 2.4. *If $\eta : \mathbb{D} \rightarrow \mathbb{C}$ is analytic then the following conditions are equivalent.*

- (1) *There exists a measure $\mu \in \mathcal{M}_\mathbb{T}$ such that $\eta = \eta_\mu$.*
- (2) *We have $\eta(0) = 0$ and $|\eta(z)| < 1$ for all $z \in \mathbb{D}$.*
- (3) *We have $|\eta(z)| \leq |z|$ for all $z \in \mathbb{D}$.*

Any measure $\mu \in \mathcal{M}_\mathbb{T}$ can be recovered from its η -transform. Indeed, the identity

$$\frac{1}{2\pi} \frac{1 + \eta_\mu(z)}{1 - \eta_\mu(z)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z} d\mu(1/\zeta),$$

whose real part is the Poisson integral of the measure $d\mu(1/\zeta)$ indicates that the family of measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ defined as

$$d\mu_\varepsilon(e^{i\theta}) = \frac{1}{2\pi} \frac{1 - |\eta_\mu(\varepsilon e^{i\theta})|^2}{|1 - \eta_\mu(\varepsilon e^{i\theta})|^2} d\theta$$

converges to ν weakly on \mathbb{T} as $\varepsilon \downarrow 0$, where $d\mu(\zeta) = d\mu(1/\zeta)$.

Recall that measures in $\mathcal{M}_\mathbb{T}^\times$ have nonzero mean. That is, if $\mu \in \mathcal{M}_\mathbb{T}^\times$ then

$$\eta'_\mu(0) = \int_{\mathbb{T}} \zeta d\mu(\zeta) \neq 0,$$

and therefore η_μ is invertible in a neighborhood of zero. This shows that the inverse η_μ^{-1} is defined for sufficiently small z , and so is

$$\Sigma_\mu(z) = \frac{\eta_\mu^{-1}(z)}{z}.$$

For $n \in \mathbb{N}$, the multiplicative free convolution power $\mu^{\boxtimes n}$ satisfies

$$\Sigma_{\mu^{\boxtimes n}}(z) = \Sigma_\mu^n(z)$$

in a neighborhood of zero. The following theorem is the generalization to and multiplicative free convolution power $t \geq 1$ [4].

Theorem 2.5. *If $\mu \in \mathcal{M}_\mathbb{T}^\times$ and $t > 1$ then the following statements hold.*

- (1) *There exists a measure $\mu_t \in \mathcal{M}_\mathbb{T}^\times$ such that $\Sigma_{\mu_t}(z) = \Sigma_\mu^t(z)$ holds in a neighborhood of zero.*

- (2) *There exists an analytic function $\omega_t : \mathbb{D} \rightarrow \mathbb{D}$ such that $|\omega_t(z)| \leq |z|$ and $\eta_{\mu_t}(z) = \eta_\mu(\omega_t(z))$ for all $z \in \Omega$.*
- (3) *The function ω_t is given by*

$$\omega_t(z) = \eta_{\mu_t}(z) \left[\frac{z}{\eta_{\mu_t}(z)} \right]^{1/t}, \quad z \in \mathbb{D}.$$

- (4) *The analytic function $\Phi_t : \mathbb{D} \rightarrow \mathbb{C}$ defined by*

$$\Phi_t(z) = z \left[\frac{z}{\eta_{\mu_t}(z)} \right]^{t-1}, \quad z \in \mathbb{D},$$

satisfies $\Phi_t(\omega_t(z)) = z$ for $z \in \mathbb{D}$.

- (5) *A point $1/\zeta$ is an atom of μ_t if $\mu(\{1/\omega_t(\zeta)\}) > (t-1)/t$, in which case we have*

$$\mu_t(1/\zeta) = t\mu(\{1/\omega_t(\zeta)\}) - (t-1).$$

Remark 2.6. For $\mu \in \mathcal{M}_{\mathbb{T}}^\times$, let $\kappa_\mu(z) = z/\eta_\mu(z)$, $z \in \mathbb{D}$. Observe that the function $\Phi_t(z) = z\kappa_\mu(z)^{t-1}$ depends on the choice of extracting roots, and therefore the measure μ_t in Theorem 2.5(1) is not unique. However, there is only one measure μ_t satisfying $\eta_{\mu_t} = \eta_\mu \circ \omega_t$ if Φ_t is chosen.

The function ω_t in the preceding result is obtained as a consequence of the following global inversion theorem [4].

Theorem 2.7. *Let $\Phi : \mathbb{D} \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic function such that $\Phi(0) = 0$ and $|z| \leq |\Phi(z)|$ for all $z \in \mathbb{D}$. Then the following statements hold.*

- (1) *There exists a continuous function $\omega : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ such that $\omega(\mathbb{D}) \subset \omega(\mathbb{D})$, $\omega|_{\mathbb{D}}$ is analytic, and $\Phi(\omega(z)) = z$ for all $z \in \mathbb{D}$.*
- (2) *If $\zeta \in \mathbb{T}$ is such that $|\omega(\zeta)| < 1$ then ω can be continued analytically to a neighborhood of ζ .*
- (3) *The set $\{z \in \mathbb{D} : |\Phi(z)| < 1\}$ is simply connected bounded by a simple closed curve. This set coincides with $\omega(\mathbb{D})$ and its boundary is $\omega(\mathbb{T})$.*
- (4) *If $z \in \omega(\overline{\mathbb{D}}) \cap \mathbb{T}$ then the entire radius $\{rz : 0 \leq r < 1\}$ is contained in $\omega(\mathbb{D})$.*

3. SUPPORT OF THE MEASURE $\mu^{\boxtimes t}$ ON \mathbb{R}_+

Throughout this section, we fix a measure $\mu \in \mathcal{M}_{\mathbb{R}_+}^\times$ and investigate the support of $\mu^{\boxtimes t}$, $t \geq 1$, which is the unique measure defined in Theorem 2.2. Let

$$\Gamma_t = \{z \in \mathcal{S}_\pi^- : H_t(z) \in \mathcal{S}_\pi\},$$

where H_t is the function defined as in (2.5). In the following proposition, we describe the set Γ_t in terms of κ_μ .

Proposition 3.1. *The set Γ_t can be expressed as*

$$(3.1) \quad \Gamma_t = \left\{ z \in \mathcal{S}_\pi^- : \frac{-\arg \kappa_\mu(\Lambda(z))}{\arg \Lambda(z)} < \frac{1}{t-1} \right\}.$$

Moreover, Γ_t is a simply connected domain whose boundary consists of two simple curves one of which is the real line.

Proof. Since $H_t(\bar{z}) = \overline{H_t(z)}$ for any $z \in \mathcal{S}_\pi$, by (2.7) we see that $\Im H_t(z) \in (t\Im z, \Im z]$ for $z \in \mathcal{S}_\pi^-$. This shows that a point $z \in \mathcal{S}_\pi^-$ satisfies $H_t(z) \in \mathcal{S}_\pi$ if and only if

$$\Im l_\mu(z) > -\frac{\pi + \Im z}{t-1} = -\frac{\arg \Lambda(z)}{t-1},$$

Since $-\kappa_\mu(\Lambda(z)) \in \mathbb{C}^+$ and $\arg[-\kappa_\mu(\Lambda(z))] = \pi + \arg \kappa_\mu(\Lambda(z))$ for $z \in \mathcal{S}_\pi^-$, the description (3.1) for Γ_t follows from (2.1). Finally, the fact $H_t(\bar{z}) = \overline{H_t(z)}$ shows that the set $\{z \in \mathcal{S}_\pi : H_t(z) \in \mathcal{S}_\pi\}$ is symmetry with respect to the real line, and hence the second assertion follows. \square

Proposition 3.2. *The function l_μ defined in (2.6) has a continuous extension to the boundary $\bar{\Gamma}_t$ and the extension is Lipschitz continuous. Consequently, the function $\kappa_\mu \circ \Lambda$ has a continuous extension to $\bar{\Gamma}_t$.*

Proof. By Theorem 2.3(2), we have

$$\left| \frac{H_t(z_1) - H_t(z_2)}{z_1 - z_2} \right| = \left| 1 + (t-1) \frac{l_\mu(z_1) - l_\mu(z_2)}{z_1 - z_2} \right| \leq 2(t+1), \quad z_1, z_2 \in \Gamma_t,$$

which yields that

$$\left| \frac{l_\mu(z) - l_\mu(z_2)}{z_1 - z_2} \right| \leq \frac{2t+3}{t-1}, \quad z_1, z_2 \in \Gamma_t.$$

This shows that l_μ extends continuously to $\bar{\Gamma}_t$ whose extension is Lipschitz continuous. Since $\kappa_\mu(\Lambda(z)) = \exp[l_\mu(z)]$ for all $z \in \mathcal{S}_\pi$, it follows that $\kappa \circ \Lambda$ extends continuously to $\bar{\Gamma}_t$. \square

The characterization and non-vanishing of the η -transform of a measure μ in $\mathcal{M}_{\mathbb{R}^+}^\times$ gives that

$$(3.2) \quad \arg \kappa_\mu(z) \in [-\pi + \arg z, 0), \quad z \in \mathbb{C}^+.$$

This implies that

$$\kappa_\mu(z) = \exp[u(z)], \quad z \in \mathbb{C}^+,$$

where u is an analytic function on Ω satisfying $u(\bar{z}) = \overline{u(z)}$ for $z \in \mathbb{C}^+$ and $u(\mathbb{C}^+) \subset \mathbb{C}^- \cup \mathbb{R}$. As a consequence the Nevanlinna representation, u can be written as

$$(3.3) \quad u(z) = a - bz + \int_0^\infty \frac{1+zs}{z-s} d\rho(s), \quad z \in \Omega,$$

where $a \in \mathbb{R}$, $b \geq 0$, and ρ is some positive Borel measure on $[0, \infty)$. In the following lemmas we provide some properties of the function u .

Lemma 3.3. *If u has the expression (3.3) then $b = 0$, and therefore*

$$\kappa_\mu(z) = \exp \left(a + \int_0^\infty \frac{1+zs}{z-s} d\rho(s) \right), \quad z \in \Omega.$$

Proof. First observe that

$$\lim_{x \rightarrow +\infty} (1 + \psi_\mu(-x)) = \lim_{x \rightarrow +\infty} \int_0^\infty \frac{d\mu(s)}{1+xs} = 0$$

by dominated convergence theorem. Moreover, for $x \geq 1$ we have

$$0 < \int_0^\infty \frac{d\mu(s)}{x(s+1)} \leq 1 + \psi_\mu(-x)$$

or, equivalently,

$$0 < \frac{1}{1 + \psi_\mu(-x)} \leq \frac{x}{c},$$

where

$$c = \int_0^\infty \frac{d\mu(s)}{s+1}$$

is a finite positive number. This implies that

$$0 \leq \lim_{x \rightarrow +\infty} \frac{\log[1 + \psi_\mu(-x)]}{-x} \leq \lim_{x \rightarrow +\infty} \frac{\log x - \log c}{x} = 0.$$

Then the above discussions and the expression

$$\log \kappa_\mu(-x) = \log x + \log[1 + \psi_\mu(-x)] - \log[-\psi_\mu(-x)], \quad x > 0,$$

yield that

$$\lim_{x \rightarrow -\infty} \frac{u(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\log \kappa_\mu(-x)}{-x} = 0.$$

Since

$$\lim_{x \rightarrow -\infty} \frac{\int_0^\infty \frac{1+xs}{x-s} d\rho(s)}{x} = \lim_{x \rightarrow +\infty} \int_0^\infty \frac{1-xs}{x(x+s)} d\rho(s) = 0$$

by dominated convergence theorem, we must have $b = 0$, as desired. \square

In the sequel, the measure ρ will be the unique measure in the Nevanlinna representation 3.3 of u .

Lemma 3.4. *For any fixed $r > 0$, the function*

$$g(r, \theta) = -\frac{\Im u(re^{i\theta})}{\theta}$$

is decreasing on $(0, \pi)$ and $\lim_{\theta \rightarrow \pi^-} g(r, \theta) = 0$. Consequently, $\arg \kappa_\mu(z) = \Im u(z) \in (-\pi + \arg z, 0]$ for $z \in \mathbb{C}^+$.

Proof. First observe that for $\theta \in (0, \pi)$ the function $g(r, \theta)$ can be expressed as

$$g(r, \theta) = \frac{r \sin \theta}{\theta} \int_0^\infty \frac{s^2 + 1}{r^2 - 2rs \cos \theta + s^2} d\rho(s)$$

by Lemma 3.3. To show that $g(r, \cdot)$ is decreasing on $(0, \pi)$, it suffices to show that it has a negative derivative. This follows from the facts

$$\frac{d}{d\theta} \left(\frac{\sin \theta}{\theta} \right) = \frac{\cos \theta}{\theta^2} (\theta - \tan \theta) < 0$$

and

$$\frac{d}{d\theta} \left(\frac{1}{r^2 - 2rs \cos \theta + s^2} \right) = \frac{-2rs \sin \theta}{(r^2 - 2rs \cos \theta + s^2)^2} < 0$$

for any $s \in [0, \infty)$ and $\theta \in (0, \pi)$. Since $u(x) \in \mathbb{R}$ for $x < 0$, it follows that $\lim_{\theta \rightarrow \pi^-} g(r, \theta) = 0$ for any $r > 0$. The last assertion follows from the above discussion, (3.2), and the continuity of u on \mathbb{C}^+ . \square

For $t > 0$, the function Φ_t defined in Theorem 2.2(3) can be expressed as

$$(3.4) \quad \Phi_t(z) = z \exp[(t-1)u(z)].$$

Then there exists an analytic function $\omega_t : \Omega \rightarrow \Omega$ satisfying the properties listed in Theorem 2.2(2). Indeed, we have the relations

$$\Phi_t = \Lambda \circ H_t \circ \Lambda^{-1} \quad \text{and} \quad \omega_t = \Lambda \circ \varpi_t \circ \Lambda^{-1}.$$

The subordination function ω_t has an important property, which is stated in the following result.

Proposition 3.5. *The function ω_t is the η -transform of some \boxtimes -infinitely divisible measure in $\mathcal{M}_{\mathbb{R}^+}^\times$.*

Proof. Since ω_t satisfies the conditions in Theorem 2.1, we must have $\omega_t = \eta_{\nu_t}$ for some measure $\nu_t \in \mathcal{M}_{\mathbb{R}^+}^\times$. It is clear that the Σ -transform of ν_t is given by $\Sigma_{\nu_t}(z) = \Phi_t(z)/z = \exp[(t-1)u(z)]$. Then the desired result follows from [Theorem 6.12, 10]. \square

Let $\mu^{\boxtimes t}$ be the unique measure in $\mathcal{M}_{\mathbb{R}^+}^\times$ such that

$$\eta_{\mu^{\boxtimes t}} = \eta_\mu \circ \omega_t.$$

Our analysis of the support of $\mu^{\boxtimes t}$ will be based on the functions $g : (0, \infty) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ and $A_t : \mathbb{R}^+ \rightarrow [0, \pi]$ which are defined as

$$g(r) = \int_0^\infty \frac{r(s^2 + 1)}{(r-s)^2} d\rho(s)$$

and

$$A_t(r) = \inf \left\{ \theta \in (0, \pi) : \frac{-\Im u(re^{i\theta})}{\theta} < \frac{1}{t-1} \right\},$$

respectively. The following set, associated with the function g ,

$$V_t^+ = \left\{ r \in (0, \infty) : g(r) > \frac{1}{t-1} \right\}$$

will also play an important role in the investigation of the support of $\mu^{\boxtimes t}$. Let

$$\Omega_t = \Lambda(\Gamma_t).$$

The following lemma provides some basic properties about the functions and set defined above.

Lemma 3.6. *Let $t > 1$. Then we have*

$$(3.5) \quad \Omega_t = \{re^{i\theta} : A_t(r) < \theta < \pi \text{ and } r \in (0, \infty)\},$$

$$(3.6) \quad \partial\Omega_t = \{re^{iA_t(r)} : r \in (0, \infty)\} \cup (-\infty, 0].$$

For any $r > 0$, we have

$$(3.7) \quad A_t(r) \in [0, \pi)$$

and

$$(3.8) \quad V_t^+ = \{r \in (0, \infty) : A_t(r) > 0\}.$$

Moreover, for any $r \in (0, \infty)$ we have

$$(3.9) \quad \lim_{\theta \downarrow A_t(r)} \frac{-\Im u(re^{i\theta})}{\theta} \leq \frac{1}{t-1},$$

where the equality holds if $r \in V_t^+$, that is,

$$(3.10) \quad \Im u(re^{iA_t(r)}) = -\frac{A_t(r)}{t-1}, \quad r \in V_t^+.$$

Proof. By Proposition 3.1 and Lemma 3.4 we see that $re^{i\theta} \in \Omega_t$ if and only if $\theta \in (0, \pi)$ and $g(r, \theta) < 1/(t-1)$. Since $g(r, \theta)$ is a decreasing function on $(0, \pi)$ for $r > 0$, it is clear that (3.5) and (3.6) hold by the definition of $A_t(t)$. Moreover, since $\lim_{\theta \rightarrow \pi^-} g(r, \theta) = 0$ for any $r > 0$, $A_t(r)$ must belong to the interval $[0, \pi)$. Next, observe that for any $r > 1$,

$$\lim_{\theta \rightarrow 0^+} g(r, \theta) = g(r)$$

by the monotone convergence theorem, and therefore the equation (3.8) holds. The inequality in (3.9) follows from the above discussion. If the strict inequality in (3.9) occurs for some $r \in V_t^+$ then it will violate the definition of $A_t(r)$, whence the proof is complete. \square

Proposition 3.7. *For any $re^{i\theta} \in \Omega_t$, the arc $\{re^{i\phi} : \theta \leq \phi < \pi\}$ is contained in Ω_t . Consequently, the set Ω_t consists of one connected component and $\partial\Omega_t$ is a simple curve.*

Proposition 3.8. *The function $u(z)$ has a continuous extension to $\overline{\Omega}_t$. Consequently, Φ_t and ω_t extend continuously to $\overline{\Omega}_t$ and $\mathbb{C}^+ \cup \mathbb{R}$, respectively.*

Proof. Since $u \circ \Lambda$ has a continuous extension to $\overline{\Gamma}_t$ by Proposition 3.8 and $\overline{\Omega}_t = \Lambda(\overline{\Gamma}_t)$, the desired result follows. \square

Lemma 3.9. *If g is bounded on some open interval I then $\rho(I) = 0$ and g is strictly convex on I . In particular, this is true if I is contained in $(V_t^+)^c$.*

Proof. Suppose that g is bounded by M on I . By countable additivity of ρ , it suffices to show that $\rho(J) = 0$ for any closed interval J contained in I . Let $c = \min\{x : x \in J\}$ and $[a, b] \subset J$. Then $g((a+b)/2) \leq M$ gives

$$M \geq \int_a^b \frac{r(s^2+1)}{(r-s)^2} d\rho(s) \geq \int_a^b \frac{c(c^2+1)}{\left(\frac{b-a}{2}\right)^2} d\rho(s) = 4c(c^2+1) \frac{\rho((a,b))}{(b-a)^2}$$

or, equivalently,

$$\frac{\rho((a,b))}{(b-a)^2} \leq \frac{M}{4c(c^2+1)} < \infty$$

since $c \neq 0$. Since $[a, b]$ can be an arbitrary subinterval in J , we conclude that $\rho(J) = 0$, as desired. \square

Observe that the mapping $r \mapsto re^{iA_t(r)}$ is a homeomorphism of $(0, \infty)$ onto $\partial\Omega_t \cap (\mathbb{C}^+ \cup (0, +\infty))$. It turns out that the function $h_t : (0, \infty) \rightarrow (0, \infty)$ defined as

$$h_t(r) = \Phi_t(re^{iA_t(r)})$$

is a homeomorphism of $(0, \infty)$. We are now at the position to introduce the main theorem of this section. For any measure ν , denote by ν^{ac} and $\text{supp}(\nu)$ the absolutely continuous part and support of ν , respectively.

Theorem 3.10. *Suppose that μ is a measure in $\mathcal{M}_{\mathbb{R}_+}^\times$ and $t > 1$. Let*

$$S_t = \left\{ \frac{1}{h_t(r)} : r \in V_t^+ \right\}.$$

Then the following statements hold.

- (1) *The measure $(\mu^{\boxtimes t})^{\text{ac}}$ is concentrated on the closure of S_t .*
- (2) *The density of $(\mu^{\boxtimes t})^{\text{ac}}$ is analytic on the set S_t and is given by*

$$\frac{d(\mu^{\boxtimes t})^{\text{ac}}}{dx} \left(\frac{1}{h_t(r)} \right) = \frac{1}{\pi} \frac{h_t(r) l_t(r) \sin \theta_t(r)}{1 - 2l_t(r) \cos \theta_t(r) + l_t^2(r)}, \quad r \in V_t^+,$$

where

$$l_t(r) = r \exp \Re u(r e^{iA_t(r)})$$

and

$$\theta_t(r) = \frac{tA_t(r)}{t-1}$$

for $r \in V_t^+$.

- (3) *The number of components in $\text{supp}(\mu^{\boxtimes t})^{\text{ac}}$ is a decreasing function of t .*

Proof. Let $z = r e^{iA_t(r)}$, $r > 0$. First claim that $\Im \eta_\mu(z) = 0$ if and only if $r \notin V_t^+$. Since $A_t(r) - \Im u(z) \in [0, \pi]$ by Lemma 3.4, the identity

$$\eta_\mu(z) = z \exp[-u(z)] = r \exp[iA_t(r) - u(z)]$$

implies that $\Im \eta_\mu(z) = 0$ if and only if $A_t(r) = \Im u(z)$ or $A_t(r) = \pi + \Im u(z)$. If $A_t(r) > 0$ then $\Im u(z) = -A_t(r)/(t-1) < 0$ by (3.10). Further suppose that $A_t(r) = \pi + \Im u(z)$. This gives $A_t(r) = (t-1)\pi/t$ and $\Im u(z) = -\pi/t$, and therefore

$$\arg \eta_\mu(z) = \arg z - \Im u(z) = \pi,$$

which is a contradiction since $\arg \eta_\mu(z) \in [A_t(r), \pi]$ by Proposition 2.1. This shows the necessity. Conversely, if $r \notin V_t^+$, i.e., $A_t(r) = 0$ by (3.8) then $u(z) \in \mathbb{R}$, and hence $\arg \eta_\mu(z) = \arg r - \Im u(r) = 0$, and hence $\Im \eta_\mu(z) = 0$.

To verify the assertions (1) and (2), observe that $\eta_{\mu^{\boxtimes t}}(h_t(r)) = (\eta_\mu \circ \omega_t)(\Phi_t(z)) = \eta_\mu(z)$ and

$$\Im \left(\frac{1}{1 - \eta_\mu(z)} \right) = \frac{\Im \eta_\mu(z)}{|1 - \eta_\mu(z)|^2} \neq 0$$

if and only if $r \in V_t^+$ by the above discussion. This implies (1) since

$$\begin{aligned} \frac{d(\mu^{\boxtimes t})^{\text{ac}}}{dx} \left(\frac{1}{h_t(r)} \right) &= \frac{1}{\pi} \Im \left(\frac{h_t(r)}{1 - \eta_{\mu^{\boxtimes t}}(h_t(r))} \right) \\ &= \frac{h_t(r)}{\pi} \Im \left(\frac{1}{1 - \eta_\mu(z)} \right) \\ &= \frac{h_t(r) \Im \eta_\mu(z)}{\pi |1 - \eta_\mu(z)|^2}. \end{aligned}$$

by Stieltjes inversion formula. The desired density for $(\mu^{\boxtimes t})^{\text{ac}}$ follows from the identities

$$A_t(r) - \Im u(z) = A_t(r) + \frac{A_t(r)}{t-1} = \theta_t(r), \quad r \in V_t^+,$$

$$\Re \eta_\mu(z) = l_t(r) \cos \theta_t(r) \quad \text{and} \quad \Im \eta_\mu(z) = l_t(r) \sin \theta_t(r).$$

The analyticity of this density wherever it is positive follows from Theorem 2.3(4). Indeed, if $r \in V_t^+$ then $\omega_t(h_t(r)) = (\omega_t \circ \Phi_t)(re^{iA_t(r)}) = re^{iA_t(r)} \in \mathbb{C}^+$, which yields that A_t is analytic in a neighborhood of r . To prove the statement (3), it is enough to show that V_t^+ is a decreasing set as t increases. This will hold if we can show that g never has a local maximum in any open interval I in V_t^+ . Indeed, the function g is strictly convex on I by Lemma 3.9, whence (3) follows. \square

Recall that a point $x \in (0, \infty)$ is an atom for μ if and only if $\eta_\mu(1/x) = 1$ and $\eta'_\mu(1/x)$ is finite, in which case we have

$$\eta'_\mu(1/x) = \frac{x}{\mu(\{x\})}.$$

Proposition 3.11. *If $r \in (0, \infty)$ and $t > 1$ then the following statements are equivalent.*

- (1) $r \in (V_t)^c$ and $\eta_\mu(r) = 1$;
- (2) $\eta_{\mu^{\boxtimes t}}(r^t) = 1$;
- (3) $\mu(\{1/r\}) \geq 1 - t^{-1}$.

If (1)-(3) hold then

$$(3.11) \quad 1 + \int_0^\infty \frac{r(s^2 + 1)}{(r - s)^2} d\rho(s) = \frac{1}{\mu(\{1/r\})}.$$

Proof. The equivalence of (2) and (3) was proved in [4]. We will show that (1) and (2) are equivalent. If (1) holds then $A_t(r) = 0$ by (3.8), and hence $e^{u(r)} = r$ and $h_t(r) = \Phi_t(r) = r \exp[(t-1)u(r)] = r^t$. This implies that $\eta_{\mu^{\boxtimes t}}(r^t) = \eta_{\mu^{\boxtimes t}}(h_t(r)) = \eta_\mu(r) = 1$ and (2) holds. Conversely, suppose that (2) holds and $h_t(s) = r^t$ for some $s \in (0, \infty)$. Then we have $\eta_\mu(se^{iA_t(s)}) = \eta_{\mu^{\boxtimes t}}(r^t) = 1$. By the fact that $\arg \eta_\mu(z) \in [\arg z, \pi]$ for any $z \in \mathbb{C}^+$, we must have $A_t(s) = 0$ and $s \in (V_t^+)^c$. Then the identities $se^{u(s)} = \eta_\mu(s) = 1$ and $s \exp[(t-1)u(s)] = \Phi_t(s) = r^t$ yield $r = s$, which implies (1). Taking the Julia-Carathéodory derivative of $\eta_\mu(z) = ze^{-u(z)}$ gives $\eta'_\mu(z) = e^{-u(z)} - zu'(z)e^{-u(z)}$, and therefore

$$r\eta'_\mu(r) = re^{-u(r)} - ru'(r)re^{-u(r)} = 1 - ru'(r).$$

On the other hand, we have

$$u'(r) = \lim_{\epsilon \rightarrow 0} \frac{u(re^{i\epsilon}) - u(r)}{re^{i\epsilon} - r} = \lim_{\epsilon \rightarrow 0} \int_0^\infty \frac{-(s^2 + 1)d\rho(s)}{(re^{i\epsilon} - s)(r - s)} = - \int_0^\infty \frac{(s^2 + 1)}{(r - s)^2} d\rho(s).$$

Hence the equation (3.11) follows from the above discussions. \square

Proposition 3.12. *Let $r \in (0, \infty)$. Then the point r^t is an atom of $\mu^{\boxtimes t}$ if and only if $\mu(\{r\}) > (t-1)/t$, in which case,*

$$\mu^{\boxtimes t}(\{r^t\}) = t\mu(\{r\}) - (t-1).$$

Proof. By Theorem 2.2(4), it suffices to show the necessity. Suppose that r^t is an atom of $\mu^{\boxtimes t}$ and let $s = 1/r$. Then by the proof of Proposition 3.11 we see that $h_t(s) = \Phi_t(s) = s^t$. Taking the Julia-Carathéodory derivative of the identity $\eta_{\mu^{\boxtimes t}} = \eta_\mu \circ \omega_t$ gives

$$\eta_{\mu^{\boxtimes t}}(s^t) = \eta'_\mu(\omega_t(s^t))\omega'_t(s^t) = \frac{\eta'_\mu(s)}{\Phi'(s)},$$

where the fact that $\Phi'_t(\omega_t(s^t))\omega'_t(s^t) = 1$. Since $\eta_{\mu^{\boxtimes t}}(s^t) = r^t/\mu^{\boxtimes t}(\{r^t\}) < \infty$, we must have $\Phi'_t(s) > 0$. Taking the Julia-Carathéodory derivative of the identity $\Phi_t(z) = z \exp[(t-1)u(z)]$ gives

$$\Phi'_t(z) = \exp[(t-1)u(z)] + (t-1)zu'(z) \exp[(t-1)u(z)],$$

which implies

$$s\Phi'_t(s) = \Phi_t(s) + (t-1)u'(s)\Phi_t(s) = \Phi_t(s)[1 + (t-1)u'(s)].$$

This shows that $1 + (t-1)su'(s) > 0$ or, equivalently,

$$su'(s) < \frac{-1}{t-1}.$$

Since we have

$$su'(s) = 1 - \frac{1}{\mu(\{r\})}$$

by the equation (3.11), it is easy to see that $\mu(\{r\}) > 1 - t^{-1}$, as desired. \square

Combining Theorem 3.10 and Proposition 3.12, we have the following result.

Theorem 3.13. *If $\mu \in \mathcal{M}_{\mathbb{R}_+}^\times$ then the following statements hold.*

- (1) *The measure $\mu^{\boxtimes t}$, $t > 1$, has at most countable many components in the support, which consists of finitely many points (atoms) and countably many arcs on \mathbb{R}_+ .*
- (2) *The number of the components in $\text{supp}(\mu^{\boxtimes t})$ is a decreasing function of t .*

4. SUPPORT OF THE MEASURE $\mu^{\boxtimes t}$ ON \mathbb{T}

Throughout this section, the measure $\mu \in \mathcal{M}_{\mathbb{T}}^\times$ is fixed. By Proposition 2.4, there exists some analytic function $u : \mathbb{D} \rightarrow \mathbb{C}$ with a nonnegative real part on \mathbb{D} . A theorem of Herglotz yields that the function u can be expressed as

$$(4.12) \quad u(z) = i\alpha + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\rho(\zeta), \quad z \in \mathbb{D}.$$

where $-\alpha = \arg |\eta'_\mu(0)| \in [-\pi, \pi]$ and ρ is a finite positive Borel measure on \mathbb{T} satisfying $\rho(\mathbb{T}) = -\log |\eta'_\mu(0)|$. For $t > 1$, let

$$\Phi_t(z) = z \exp[(t-1)u(z)], \quad z \in \mathbb{D}.$$

By Proposition 2.7, there exists a continuous function $\omega_t : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ such that $\omega_t|_{\mathbb{D}}$ is analytic, $\omega_t(0) = 0$, and $\Phi_t(\omega_t(z)) = z$ for all $z \in \mathbb{D}$. Let

$$\Omega_t = \{z \in \mathbb{D} : |\Phi_t(z)| < 1\}.$$

Then $\omega_t(\Phi_t(z)) = z$, $z \in \Omega_t$. The following proposition states another important property of ω_t .

Proposition 4.1. *The function ω_t is the η -transform of some \boxtimes -infinitely divisible measure in $\mathcal{M}_{\mathbb{T}}^{\times}$.*

Proof. The function ω_t satisfies the conditions in Theorem 2.4, and therefore it must be of the form $\omega_t = \eta_{\nu_t}$ for some measure ν_t in $\mathcal{M}_{\mathbb{T}}^{\times}$. Next, observe that the Σ -transform of ν_t is given by $\Sigma_{\nu_t}(z) = \Phi_t(z)/z = \exp[(t-1)u(z)]$. This implies the desired result by [Theorem 7.5, 9]. \square

Let $\mu^{\boxtimes t}$ be the unique measure in $\mathcal{M}_{\mathbb{T}}$ satisfying

$$(4.13) \quad \eta_{\mu^{\boxtimes t}}(z) = \eta_{\mu}(\omega_t(z)), \quad z \in \mathbb{D}.$$

Since $\omega'_t(0) = 1/\Phi'_t(\omega_t(0)) = 1/\Phi'_t(0) = \exp[-(t-1)u(0)]$, we see that

$$\eta'_{\mu^{\boxtimes t}}(0) = \eta'_{\mu}(0)\omega'_t(0) = \exp[-tu(0)],$$

which particularly shows that $\mu^{\boxtimes t} \in \mathcal{M}_{\mathbb{T}}^{\times}$.

In the rest of this section, we will investigate the support of $\mu^{\boxtimes t}$, which is the measure satisfying the requirement (4.13). Our analysis will be based on the following functions $g : [-\pi, \pi] \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ and $R_t : [-\pi, \pi] \rightarrow [0, 1]$, which are defined as

$$(4.14) \quad g(\theta) = \int_{-\pi}^{\pi} \frac{d\rho(e^{i\phi})}{1 - \cos(\theta - \phi)}$$

and

$$(4.15) \quad R_t(\theta) = \sup \left\{ r \in (0, 1) : \int_{-\pi}^{\pi} T(r, \theta - \phi) d\rho(e^{i\phi}) < \frac{1}{t-1} \right\},$$

respectively, where

$$(4.16) \quad T(r, \theta) = \frac{r^2 - 1}{\log r} \frac{1}{1 - 2r \cos \theta + r^2}$$

is a continuous function from $(0, 1) \times [-\pi, \pi]$ to \mathbb{R}^+ . The following set, associated with the function g ,

$$V_t^+ = \left\{ \theta \in [-\pi, \pi] : g(\theta) > \frac{1}{t-1} \right\},$$

will also play an important role in the study of the support of $\mu^{\boxtimes t}$.

The following lemmas provide some basic properties about the functions and set introduced above.

Lemma 4.2. *For any $\theta \in [-\pi, \pi]$, the function $T(\cdot, \theta)$ defined as in (4.16) is strictly increasing on $(0, 1)$. Consequently,*

$$T(1, \theta) = \lim_{r \uparrow 1} T(r, \theta) = \frac{1}{1 - \cos \theta}$$

for $\theta \neq 0$ and $T(1, 0) = \lim_{r \uparrow 1} T(r, 0) = +\infty$.

Proof. First note that the function

$$r \mapsto \frac{(r+1)^2}{1-2r\cos\theta+r^2}, \quad r \in (0,1),$$

is increasing since it has a non-negative derivative on $(0,1)$. Next, by some simple manipulations we see that the function

$$f(x) = \frac{e^x - 1}{x(e^x + 1)}$$

has a strictly negative derivative on $(0, \infty)$. This implies that the function

$$r \mapsto f(-\log r) = \frac{r-1}{(r+1)\log r}$$

is strictly increasing on $(0,1)$. Then the desired conclusion follows from the expression

$$T(r, \theta) = \frac{r-1}{(r+1)\log r} \frac{(r+1)^2}{1-2r\cos\theta+r^2}.$$

This completes the proof. \square

Lemma 4.3. *Let $t > 1$ and $T(r, \theta)$ be the function defined in (4.16) and Lemma 4.2 on $(0,1] \times [-\pi, \pi]$. Then*

$$(4.17) \quad \Omega_t = \{re^{i\theta} : r \in [0, R_t(\theta)) \text{ and } \theta \in [-\pi, \pi]\}$$

and

$$(4.18) \quad \partial\Omega_t = \{R_t(\theta)e^{i\theta} : \theta \in [-\pi, \pi]\}.$$

For any $\theta \in [-\pi, \pi]$ we have

$$(4.19) \quad R_t(\theta) \in (0, 1]$$

and

$$(4.20) \quad V_t^+ = \{\theta \in [-\pi, \pi] : R_t(\theta) < 1\}.$$

Moreover, for any $\theta \in [-\pi, \pi]$, we have

$$(4.21) \quad \int_{-\pi}^{\pi} T(R_t(\theta), \theta + \phi) d\rho(e^{i\phi}) \leq \frac{1}{t-1},$$

where the equality holds if $\theta \in V_t^+$.

Proof. First observe that a point $re^{i\theta}$ belongs to $\Omega_t \setminus \{0\}$ if and only if

$$g(r, \theta) := \int_{-\pi}^{\pi} T(r, \theta - \phi) d\rho(e^{i\phi}) < 1/(t-1).$$

Indeed, simple computations give

$$\log |\Phi_t(re^{i\theta})| = (\log r)[1 - (t-1)g(r, \theta)].$$

This implies that $re^{i\theta} \in \mathbb{D} \setminus \{0\}$ belongs to Ω_t if and only if $\log |\Phi_t(re^{i\theta})| < 0$, which happens if and only if $(\log r)[1 - (t-1)g(r, \theta)] < 0$ or, equivalently, $g(r, \theta) < 1/(t-1)$. By Lemma 4.2 and the definition of the function R_t we see that the identity (3.5) holds. The identity (4.18) follows directly from (4.17).

For any $\theta \in [-\pi, \pi]$ and small $\epsilon > 0$, by the fact (4.17) we have $g(R_t(\theta) - \epsilon, \theta) < 1/(t-1)$, which gives (4.21) by letting $\epsilon \rightarrow 0$ and monotone convergence theorem. This also shows that $\theta \in V_t^+$ if $R_t(\theta) < 1$. If $R_t(\theta) = 1$ then $g(r, e^{i\theta}) \leq 1/(t-1)$ for $r \in (0, 1)$, whence $g(\theta) = \lim_{r \uparrow 1^-} g(r, e^{i\theta}) \leq 1/(t-1)$. Hence the identity (4.20) holds. By the above discussion, we see that the equality in (4.21) holds if $\theta \in V_t^+$. \square

Due to the preceding lemma, we have the the following proposition which generalizes the statement (3) in Theorem 2.7.

Proposition 4.4. *For any $z \in \Omega_t$, the line segment joining the origin and z is contained in Ω_t . Consequently, the set Ω_t consists of one connected component.*

Proposition 4.5. *The function $u(z)$ has a continuous extension to $\overline{\Omega}_t$. Moreover, the extension is Lipschitz continuous on $\overline{\Omega}_t$ and can be expressed as*

$$(4.22) \quad u(z) = i\alpha + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\rho(\zeta), \quad \zeta \in \overline{\Omega}_t.$$

Proof. We first show that the integral in (4.22) converges for $z \in \overline{\Omega}_t$. It suffices to consider the case $z = e^{i\theta} \in \partial\Omega_t \cap \mathbb{T}$. Note that $z_r := rz \in \Omega_t$, $0 < r < 1$, and

$$\int_{\mathbb{T}} \frac{\zeta + z_r}{\zeta - z_r} d\rho(\zeta) = \int_{\mathbb{T}} \frac{1 - r^2}{|\zeta - z_r|^2} d\rho(\zeta) + 2i \int_{-\pi}^{\pi} \frac{r \sin(\theta - \phi) d\rho(e^{i\phi})}{1 - 2r \cos(\theta - \phi) + r^2}.$$

The first integral on the right hand of the above expression tends to zero as $r \rightarrow 1^-$ since

$$\int_{\mathbb{T}} \frac{1 - r^2}{|\zeta - z_r|^2} d\rho(\zeta) \leq \frac{-\log r}{t-1}$$

by (4.21). By the fact that the mapping

$$r \mapsto \frac{r}{1 - 2r \cos \phi + r^2}$$

is strictly increasing on $(0, 1)$ for any ϕ , we conclude that the integral

$$\int_{-\pi}^{\pi} \frac{r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} d\rho(e^{i\phi}) \rightarrow \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin(\theta - \phi)}{1 - \cos(\theta - \phi)} d\rho(e^{i\phi})$$

converges as $r \rightarrow 1^-$ by monotone convergence theorem. Since $g(\theta) \leq 1/(t-1)$ by (4.21), it follows that u has a continuous extension to $\overline{\Omega}_t$. Next, we show that this extension is Lipschitz continuous. Notice that for any different points z_1 and z_2 in Ω_t we have

$$\begin{aligned} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|} &= 2 \left| \int_{\mathbb{T}} \frac{\zeta d\rho(\zeta)}{(\zeta - z_1)(\zeta - z_2)} \right| \\ &\leq 2 \left(\int_{\mathbb{T}} \frac{d\rho(\zeta)}{|\zeta - z_1|^2} \right)^{1/2} \left(\int_{\mathbb{T}} \frac{d\rho(\zeta)}{|\zeta - z_2|^2} \right)^{1/2}, \end{aligned}$$

where the Hölder inequality is used in the the last inequality. Let ϵ be small enough so that the set $\overline{\Omega}_{t,\epsilon} := \overline{\Omega}_t \cap \{z : |z| \geq \epsilon\} \neq \emptyset$ (note that V_t^+ is an open set containing the origin). Then the fact that the function $r \mapsto \log r/(r^2 - 1)$ is decreasing on $(0, 1)$ shows that

$$\int_{\mathbb{T}} \frac{d\rho(\zeta)}{|\zeta - z|^2} \leq \frac{\log |z|}{|z|^2 - 1} \frac{1}{t-1} \leq \frac{\log \epsilon}{\epsilon^2 - 1} \frac{1}{t-1}, \quad z \in \overline{\Omega}_{t,\epsilon}.$$

On the other hand, if $z \in \Omega_t$ with $|z| \leq \epsilon$ then

$$\int_{\mathbb{T}} \frac{d\rho(\zeta)}{|\zeta - z|^2} \leq \int_{\mathbb{T}} \frac{d\rho(\zeta)}{(1 - \epsilon)^2} = (1 - \epsilon)^{-2} \rho(\mathbb{T}).$$

The above discussions yield the desired result. \square

For any $-\pi \leq a < b \leq \pi$, let $A_{a,b} = \{e^{i\theta} : a < \theta < b\}$ be an arc contained in \mathbb{T} .

Lemma 4.6. *If the function g is bounded on some open interval (a, b) then $\rho(A_{a,b}) = 0$ and g_t is strictly convex on (a, b) . In particular, this is true if (a, b) is contained in $(V_t^+)^c$.*

Proof. Suppose that g is bounded by M on (a, b) . Let (c, d) be any subinterval of (a, b) . If $\theta = (c + d)/2$ then we have

$$M \geq \int_{-\pi}^{\pi} \frac{d\rho(e^{i\phi})}{1 - \cos(\theta - \phi)} \geq \int_c^d \frac{d\rho(e^{i\phi})}{1 - \cos(\theta - \phi)} = \int_c^d \frac{d\rho(e^{i\phi})}{2 \sin^2\left(\frac{\theta - \phi}{2}\right)}.$$

Since $\sin^2 \phi \leq \phi^2$ for any $\phi \in [-\pi, \pi]$, we deduce that

$$M \geq 2 \int_a^b \frac{d\rho(e^{i\phi})}{(\theta - \phi)^2} \geq \frac{8\rho(A_{c,d})}{(d - c)^2}$$

or, equivalently,

$$\frac{\rho(A_{c,d})}{d - c} \leq 8M(d - c).$$

Since the above inequality holds for any subinterval contained in (a, b) , we conclude that the desired result holds. The strict positivity of the second order derivative

$$g''(\theta) = \frac{3}{4} \int_{[-\pi, \pi] \setminus (a, b)} \frac{d\rho(e^{i\theta})}{\sin^4\left(\frac{\theta - \phi}{2}\right)}$$

on I yield the second assertion. \square

Observe that the function $\theta \mapsto R_t(\theta)e^{i\theta}$ is a homeomorphism from $[-\pi, \pi]$ onto $\partial\Omega_t$. Since Φ_t has a continuous extension to $\overline{\Omega}_t$, we deduce that the function

$$h_t(e^{i\theta}) = \Phi_t(R_t(\theta)e^{i\theta})$$

is a homeomorphism of \mathbb{T} . Now, we are in a position to state the main theorem of this section.

Theorem 4.7. *Suppose that μ is a measure in $\mathcal{M}_{\mathbb{T}}^{\times}$ and $t > 1$. Let*

$$S_t = \{\overline{h_t(e^{i\theta})} : \theta \in V_t^+\}.$$

Then the following statements hold.

- (1) *The measure $(\mu^{\boxtimes t})^{\text{ac}}$ is concentrated on the closure of S_t .*
- (2) *The density of $(\mu^{\boxtimes t})^{\text{ac}}$ is analytic on the set S_t and is given by*

$$\frac{d\mu^{\boxtimes t}}{d\zeta} \left(\overline{h_t(e^{i\theta})} \right) = \frac{1}{2\pi} \frac{1 - l_t^2(\theta)}{1 - 2l_t(\theta) \cos \alpha(\theta) + l_t^2(\theta)}, \quad \theta \in V_t^+,$$

where

$$l_t(\theta) = R_t^{\frac{t}{t-1}}(\theta)$$

$$\alpha(\theta) = \theta - \Im u(R_t(\theta)e^{i\theta}).$$

(3) *The number of components in $\text{supp}(\mu^{\boxtimes t})^{\text{ac}}$ is a decreasing function of t .*

Proof. Let $z = R_t(\theta)e^{i\theta}$. We claim that $|\eta_\mu(z)| = 1$ if and only if $\theta \notin V_t^+$. Observe that if $\theta \in (V_t^+)^c$ then $R_t(\theta) = 1$ and $\Re u(e^{i\theta}) = 0$, which gives

$$|\eta_\mu(e^{i\theta})| = |e^{i\theta}e^{-\Re u(e^{i\theta})}| = 1.$$

If $\theta \in V_t^+$ then

$$\Re u(z) = \frac{-\log R_t(\theta)}{t-1}$$

by (4.21), whence we have

$$|\eta_\mu(z)| = R_t^{\frac{t}{t-1}}(\theta)$$

and

$$\Re \eta_\mu(z) = R_t^{\frac{t}{t-1}}(\theta) \cos(\theta - \Im u(z)).$$

Since $\eta_{\mu^{\boxtimes t}}$ is continuous on \mathbb{T} , it follows that $(\mu^{\boxtimes t})^{\text{ac}}$ is concentrated on the closure of the set of points $\zeta \in \mathbb{T}$ such that

$$\Re \left(\frac{1 + \eta_{\mu^{\boxtimes t}}(\zeta)}{1 - \eta_{\mu^{\boxtimes t}}(\zeta)} \right)$$

is finite and nonzero. Now, note that we have $\eta_{\mu^{\boxtimes t}}(h_t(e^{i\theta})) = \eta_\mu(z)$ and

$$\Re \left(\frac{1 + \eta_{\mu^{\boxtimes t}}(h_t(e^{i\theta}))}{1 - \eta_{\mu^{\boxtimes t}}(h_t(e^{i\theta}))} \right) = \Re \left(\frac{1 + \eta_\mu(z)}{1 - \eta_\mu(z)} \right) = \frac{1 - |\eta_\mu(z)|^2}{1 - 2\Re \eta_\mu(z) + |\eta_\mu(z)|^2}.$$

Since

$$\frac{d(\mu^{\boxtimes t})^{\text{ac}}}{d\zeta} \left(\overline{h_t(e^{i\theta})} \right) = \Re \left(\frac{1 + \eta_{\mu^{\boxtimes t}}(h_t(e^{i\theta}))}{1 - \eta_{\mu^{\boxtimes t}}(h_t(e^{i\theta}))} \right),$$

the desired results in (1) and (2) follow. To verify the statement (3), it suffices to show that the number of components in V_t^+ is nonincreasing as t increases. This will hold if we show that g never has a local maximum in any open interval (a, b) in V_t^+ . Indeed, the function g is strictly convex on such an interval by Lemma 4.6, whence (3) follows. \square

For the rest of this section, we discuss the atoms of $\mu^{\boxtimes t}$.

Proposition 4.8. *If $\theta \in [-\pi, \pi]$ and $t > 1$ then the following statements are equivalent.*

- (1) $\theta \in (V_t^+)^c$ and $\eta_\mu(e^{i\theta}) = 1$;
- (2) $\eta_{\mu^{\boxtimes t}}(h_t(e^{i\theta})) = 1$;
- (3) $\mu(\{e^{-i\theta}\}) \geq 1 - t^{-1}$.

If (1)-(3) hold then

$$(4.23) \quad 1 + \int_{-\pi}^{\pi} \frac{d\rho(e^{i\phi})}{1 - \cos(\theta - \phi)} = \frac{1}{\mu(\{e^{i\theta}\})}.$$

In addition, if $t \geq 2$ then the condition $\eta_{\mu^{\boxtimes t}}(e^{it\theta}) = 1$ with $|t\theta| < \pi$ is equivalent to conditions (1)-(3).

Proof. The equivalence of (2) and (3) was proved in [4]. We will show that (1) and (2) are equivalent. If (1) holds then $R_t(\theta) = 1$ by (4.19), and hence $\eta_{\mu \boxtimes t}(h_t(e^{i\theta})) = \eta_{\mu}(\omega_t(\Phi_t(e^{i\theta}))) = \eta_{\mu}(e^{i\theta}) = 1$, which yields (2). Conversely, the condition $\eta_{\mu}(R_t(\theta)e^{i\theta}) = \eta_{\mu \boxtimes t}(h_t(e^{i\theta})) = 1$ in (2) along with the fact that $|\eta_{\mu}(z)| \leq |z|$ for any $z \in \mathbb{D}$ shows $R_t(\theta) = 1$, and therefore (1) holds. This also particularly implies that

$$(4.24) \quad \exp[u(e^{i\theta})] = e^{i\theta}.$$

Next, suppose that $\eta_{\mu}(e^{i\theta}) = 1$ and $e^{i\theta}\eta'_{\mu}(e^{i\theta}) < \infty$, where $\eta'_{\mu}(e^{i\theta})$ is the Julia-Carathéodory derivative of η_{μ} at $e^{i\theta}$. Taking the Julia-Carathéodory derivative of the identity $\eta_{\mu}(z) = z \exp[-u(z)]$ gives

$$\eta'_{\mu}(z) = \exp[-u(z)] - zu'(z) \exp[-u(z)],$$

and therefore we have

$$\begin{aligned} e^{i\theta}\eta'_{\mu}(e^{i\theta}) &= e^{i\theta} \exp[-u(e^{i\theta})] - e^{i\theta}u'(e^{i\theta})e^{i\theta} \exp[-u(e^{i\theta})] \\ &= 1 - e^{i\theta}u'(e^{i\theta}). \end{aligned}$$

On the other hand, using the identity

$$\frac{u(re^{i\theta}) - u(e^{i\theta})}{(r-1)e^{i\theta}} = \int_{\mathbb{T}} \frac{2\xi d\rho(\xi)}{(\xi - re^{i\theta})(\xi - e^{i\theta})}, \quad 0 < r < 1,$$

it is easy to see that the Julia-Carathéodory derivative $u'(e^{i\theta})$ is given by

$$u'(e^{i\theta}) = \int_{\mathbb{T}} \frac{2\xi d\rho(\xi)}{(\xi - e^{i\theta})^2}.$$

Then

$$\begin{aligned} e^{i\theta}u'(e^{i\theta}) &= \Re \left(\int_{\mathbb{T}} \frac{2e^{i\theta}\xi}{(\xi - e^{i\theta})^2} d\rho(\xi) \right) \\ &= \int_{\mathbb{T}} \Re \left(\frac{2e^{i\theta}\xi}{(\xi - e^{i\theta})^2} \right) d\rho(\xi) \\ &= - \int_{-\pi}^{\pi} \frac{d\rho(e^{i\phi})}{1 - \cos(\theta - \phi)}, \end{aligned}$$

which gives the identity (4.23).

Finally, if (2) holds and $t \geq 2$ then

$$|u(e^{i\theta})| = \left| \int_{\mathbb{T}} \frac{2\Im(e^{i\theta}\bar{\xi})}{|\xi - e^{i\theta}|^2} d\rho(\xi) \right| \leq \int_{-\pi}^{\pi} \frac{d\rho(e^{i\phi})}{1 - \cos(\theta - \phi)} \leq \frac{1}{t-1} < \pi,$$

which shows that $u(e^{i\theta}) = i\theta$ by the equation (4.24). Therefore

$$h_t(e^{i\theta}) = \Phi_t(e^{i\theta}) = e^{i\theta} \exp[(t-1)u(e^{i\theta})] = e^{it\theta}$$

and $|t\theta| = |tu(e^{i\theta})| \leq t/(t-1) < \pi$. Now, suppose $\eta_{\mu \boxtimes t}(e^{it\theta}) = 1$ with $|t\theta| < \pi$ and $h_t(e^{i\phi}) = e^{it\theta}$ for some $\phi \in [-\pi, \pi]$. Then the preceding argument indicates that $u(e^{i\phi}) = i\phi$, $|\phi| \leq 1/(t-1)$ and $h_t(e^{i\phi}) = e^{it\phi}$. Since $e^{it\theta} = e^{it\phi}$ and $|t\theta|, |t\phi| \leq \pi$, we must have $\theta = \phi$. This completes the proof. \square

Proposition 4.9. *A point $1/\zeta$ is an atom of $\mu^{\boxtimes t}$ if and only if $\mu(\{1/\omega_t(\zeta)\}) > 1 - t^{-1}$, in which case we have*

$$\mu^{\boxtimes t}(\{1/\zeta\}) = t\mu(\{1/\omega_t(\zeta)\}) - (t - 1).$$

Proof. By Theorem 2.5(5), it suffices to show the necessity. Suppose that $1/\zeta$ is an atom of $\mu^{\boxtimes t}$ and let $h_t(e^{i\theta}) = \xi$ for some $\theta \in [-\pi, \pi]$. Then by Proposition 3.11 we see that $\Phi_t(e^{i\theta}) = \xi$. Taking the Julia-Carathéodory derivative of the identity $\eta_{\mu^{\boxtimes t}} = \eta_\mu \circ \omega_t$ gives

$$\xi \eta'_{\mu^{\boxtimes t}}(\xi) = \xi \eta'_\mu(\omega_t(\xi)) \omega'_t(\xi) = e^{i\theta} \eta'_\mu(e^{i\theta}) \frac{\xi}{e^{i\theta} \Phi'_t(e^{i\theta})},$$

where the fact that $\Phi'_t(\omega_t(\xi)) \omega'_t(\xi) = 1$. Since $\xi \eta'_{\mu^{\boxtimes t}}(\xi) = 1/\mu^{\boxtimes t}(\{1/\xi\}) < \infty$, we must have

$$(4.25) \quad \frac{\xi}{e^{i\theta} \Phi'_t(e^{i\theta})} < \infty.$$

Taking the Julia-Carathéodory derivative of the identity $\Phi_t(z) = z \exp[(t-1)u(z)]$ gives

$$\Phi'_t(z) = \exp[(t-1)u(z)] + (t-1)u'(z)z \exp[(t-1)u(z)],$$

which implies

$$e^{i\theta} \Phi'_t(e^{i\theta}) = \Phi_t(e^{i\theta}) + (t-1)e^{i\theta} u'(e^{i\theta}) \Phi_t(e^{i\theta}) = \xi[1 + (t-1)e^{i\theta} u'(e^{i\theta})].$$

Using the above identity, the condition (4.25) is equivalent to

$$e^{i\theta} u'(e^{i\theta}) > \frac{-1}{t-1}.$$

Since we have

$$e^{i\theta} u'(e^{i\theta}) = 1 - \frac{1}{\mu(\{e^{-i\theta}\})}$$

by Proposition 3.11, it is easy to see that $\mu(\{e^{-i\theta}\}) > 1 - t^{-1}$, as desired. \square

Combining Proposition 4.9 and Theorem 4.7, we have the following result.

Theorem 4.10. *If $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$ then the following statements hold.*

- (1) *The measure $\mu^{\boxtimes t}$, $t > 1$, has at most countable many components in the support, which consists of finitely many points (atoms) and countably many arcs on \mathbb{T} .*
- (2) *The number of the components in $\text{supp}(\mu^{\boxtimes t})$ is a decreasing function of t .*

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